

Two Person Games (Setting up the Pay-off Matrix)

Mathematical Game theory was developed as a model of situations of conflict. Such situations and interactions will be called **games** and they have participants who are called players. We will focus on games with **exactly two players**. These two players compete for a **payoff** that one player pays to the other. These games are called **zero-sum games** because one player's loss is the other player's gain and the payoff to both players for any given scenario adds to zero.

Example: Coin Matching Game Roger and Colleen play a game. Each one has a coin. They will both show a side of their coin simultaneously. If both show heads, no money will be exchanged. If Roger shows heads and Colleen shows tails then Colleen will give Roger \$1. If Roger shows tails and Colleen shows heads, then Roger will pay Colleen \$1. If both show tails, then Colleen will give Roger \$2.

This is a **Two person game**, the players are Roger and Colleen. It is also a **zero-sum game**. This means that Roger's gain is Colleen's loss.

We can use a 2×2 array or matrix to show all four situations and the results as follows:

		Colleen	
		Heads	Tails
Roger	Heads	Roger pays \$0 Colleen pays \$0	Roger gets \$1 Colleen pays \$1
	Tails	Roger pays \$1 Colleen gets \$1	Roger gets \$2 Colleen pays \$2

This is called a **two-person, zero-sum game** because the amount won by each player is equal to the negative of the amount won by the opponent for any given situation. The amount won by either player in any given situation is called the **pay-off** for that player. A negative pay-off denotes a loss of that amount for the player. Since it is a zero-sum game, we can deduce the pay-off of one player from that of the other, thus we can deduce all of the above information from the **pay-off matrix** shown below. The **pay-off matrix for a game** shows only the pay-off for the row player for each scenario.

		Colleen	
		<i>H</i>	<i>T</i>
R	o	0	1
g.	<i>T</i>	-1	2

A player's plan of action against the opponent is called a strategy. In the above example, each player has two possible strategies; H and T. We will try to determine each player's best strategy assuming both players want to maximize their pay-off. Sometimes our conclusions will make most sense when we consider players who are repeatedly playing the same game.

Assumptions

- We will limit our attention to **Two person zero-sum games** in this course.
- We will make the added assumption that each player is striving to maximize their pay-off.

Although one can envisage situations of conflict or interaction where these assumptions do not apply, models for these situations are beyond the scope of our course. However often one can throw light on many situations with the above assumptions.

Each Player is assumed to have several **options or strategies** that he/she can exercise. We have two further assumptions concerning the player's options.

- Each time the game is played, each player selects one option.
- The players decide on their options simultaneously and independently of one another.
- Each player has full knowledge of the strategies available to himself and his opponent and the pay-offs associated to each possible scenario. (However neither player knows which strategy their opponent will choose.)

Again these are simplifying assumptions, but nevertheless can help greatly in the development of a rewarding strategy.

Pay-Off Matrix: In the general situation for a two-player, zero sum game, we will call the two players R (for row) and C (for column). For each such game, we can represent all of the information about the game in a matrix. This matrix is called the **Pay-off matrix for R** . It is a matrix with a list of R 's strategies as labels for the rows and a list of C 's strategies as labels for the columns. The entries in the pay-off matrix are what R gains for each combination of strategies. If this is a negative number than it represents a loss for R .

Example Consider the coin matching game played by Roger and Colleen described above. What is the payoff for Roger if Roger shows heads and Colleen shows tails? What is the pay-off for Colleen in this situation?

Example (Two Finger Morra) Ruth and Charlie play a game. At each play, Ruth and Charlie simultaneously extend either one or two fingers and call out a number. The player whose call equals the total number of extended fingers wins that many pennies from the opponent. In the event that neither player's call matches the total, no money changes hands.

(a) Write down a pay-off matrix for this game (here the strategy (1, 2) means that the player holds up one finger and shouts 2).

		Charlie			
		(1, 2)	(1, 3)	(2, 3)	(2, 4)
R u t h	(1, 2)				
	(1, 3)				
	(2, 3)				
	(2, 4)				

		Charlie			
		(1, 2)	(1, 3)	(2, 3)	(2, 4)
R	(1, 2)	0	2	-3	0
u	(1, 3)	-2	0	0	3
t	(2, 3)	3	0	0	-4
h	(2, 4)	0	-3	4	0

(b) What is the payoff for Ruth if Ruth shows two fingers and calls out 4 and Charlie shows 1 finger and calls out 3? What is the payoff for Charlie in this situation?

(2, 4) for Ruth is row 4 and (1, 3) for Charlie is column 2. Hence -3 is Ruth's payoff so she gives Charlie 3 cents or Charlie's payoff is 3.

(c) Play this game 5 times with the person next to you and record the strategies and the payoff for both players below. (One of you assumes the persona of Ruth and the other Charlie for a bit.)

Ruth		Charlie	
strategy	pay-off	strategy	pay-off

Example In the game of Rock-scissors-paper, the players face each other and simultaneously display their hands in one of the three following shapes: a fist denoting a rock (R), the forefinger and middle finger extended and spread so as to suggest scissors (S), or a downward facing palm denoting a sheet of paper (P). The rock wins over the scissors since it can shatter them, the scissors wins over the paper since they can cut it, and the paper wins over the rock since it can be wrapped around it. The winner collects a penny from the opponent and no money changes hands in the case of a tie. What is the pay-off matrix for this game? (Use R, S, and P to denote the strategies Rock, Scissors and Paper respectively.)

	<i>R</i>	<i>P</i>	<i>S</i>
<i>R</i>	0	-1	1
<i>P</i>	1	0	-1
<i>S</i>	-1	1	0

Example: Football Run or Pass? [Winston] (Using averages as payoffs) In football, the offense selects a play and the defense lines up in a defensive formation. We will consider a very simple model of play selection in which the offense and defense simultaneously select their play. The offense may choose to run or to pass and the defense may choose a run or a pass defense. One can use the average yardage gained or lost in this particular League as payoffs and construct a payoff matrix for this two player zero-sum game. Lets assume that if the offense runs and the defense makes the right call, yards gained average out at a loss of 5 yards for the offense. On the other hand if offense runs and defense makes the wrong call, the average gain is 5 yards. On a pass, the right defensive call usually results in an incomplete pass averaging out to a zero yard gain for offense and the wrong defensive call leads to a 10 yard gain for offense. Set up the payoff matrix for this zero-sum game.

		Defense	
		Run Defense	Pass Defense
Offense	Run		
	Pass		

	<i>DR</i>	<i>DP</i>
<i>OR</i>	-5	5
<i>OP</i>	10	0

Constant-Sum Games

In some games, we have the same assumptions as above except that the pay-offs of both players add to a constant, for example if both players are competing for a share of a market of fixed size, we can write pay-offs as percentage of the market for each player with the percentages adding to 100. All results and methods that we study for zero-sum games also work for constant sum games.

Example (Using Percentages or proportions) Rory and Corey own stores next to each other. Each day they announce a sale giving 10% or 20% off. If they both give 10% off then Rory gets 70% of the customers. If Rory announces a 10% sale and Corey announces a 20% sale, then Rory gets 30% of the customers. If Rory announces a 20% off sale and Corey a 10% off sale, then Rory gets 90% of the customers. Finally if they both announce a 20% off sale, Rory gets 50% of the customers. Represent this in a payoff matrix, assuming that between them Rory and Corey get all of the customers each day and each customer patronizes only one of the shops each day.

Suppose we denote the two choices by $BS = 20\%$ off and $SS = 10\%$ off (big sale verses small sale). Let us write the payoff matrix for Rory (that is Rory's strategy is the left hand column and Corey's strategy is the top row).

	BS	SS
BS	0.5	0.9
SS	0.3	0.7

Example (Using Probabilities as pay-offs) General Roadrunner and General Coyote are generals of opposing armies. Every day General Roadrunner sends out a bombing sortie consisting of a heavily armed bomber plane and a lighter support plane. The sortie's mission is to drop a single bomb on General Coyote's forces. However a fighter plane of General Coyote's army is waiting for them in ambush and it will dive down and attack one of the planes in the sortie once. The bomber has an 80% chance of surviving such an attack, and if it survives it is sure to drop the bomb right on the target. General Roadrunner also has the option of placing the bomb on the support plane. In this case, due to this plane's lighter armament and lack of proper equipment, the bomb will reach its target with a probability of only 50% or 90%, depending on whether or not it is attacked by General Coyote's fighter. Represent this information on a pay-off matrix for General Roadrunner.

Coyote's strategies are attack the bomber B or attack the fighter F. Roadrunner's are to put the bomb on the bomber B or the fighter F. If you like you can use different letters for Roadrunner's strategies.

	<i>B</i>	<i>F</i>
<i>B</i>	0.8	1.0
<i>F</i>	0.9	0.5

Extras

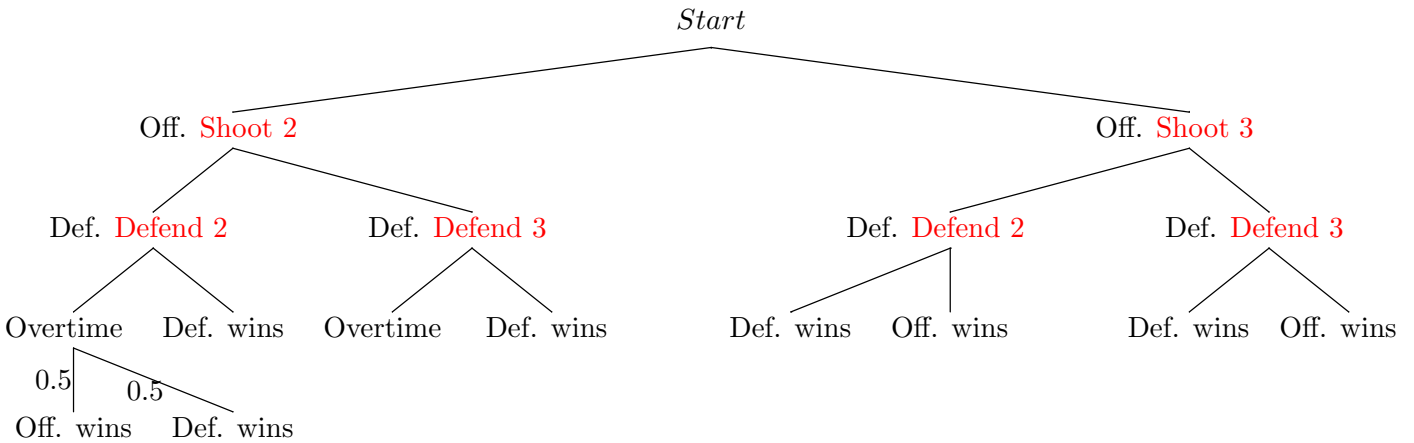
Using probabilities as payoffs

Endgame Basketball [Ruminski] Often in late game situations, a team with the ball may find themselves down by two points with the shot clock turned off. In this situation, the offensive team must decide whether to shoot for two points, hoping to tie the game and win in overtime, or to try for a three pointer and win the game without overtime. The defending team must decide whether to defend the inside or outside shot. We assume that the probability of winning in overtime is 50% for both teams.

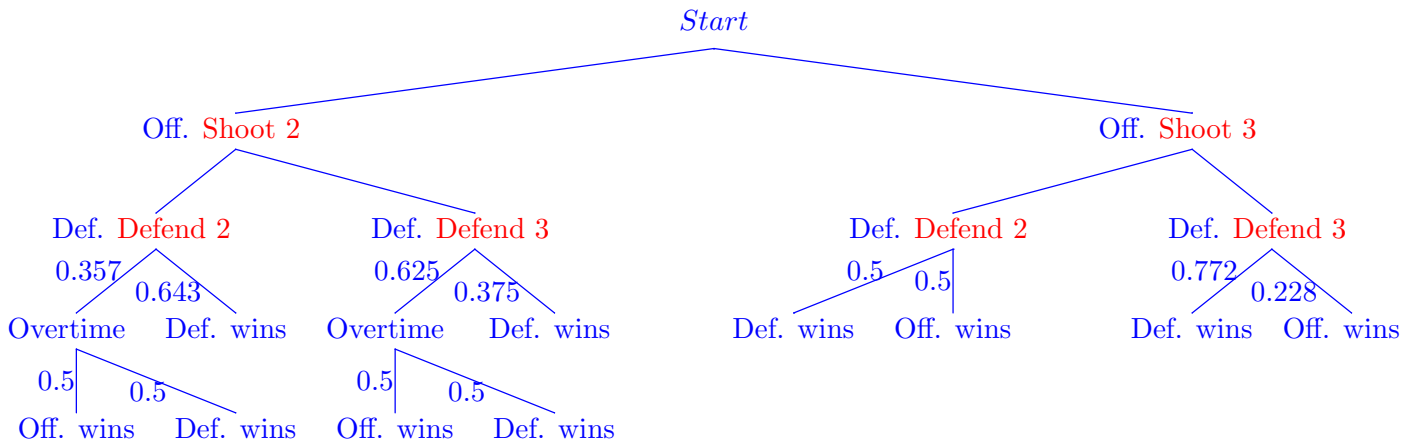
In this situation, the offensive team's coach will ask for a timeout in order to set up the play. Simultaneously, the defensive coach will decide how to set up the defense to ensure a win. Therefore we can consider this as a simultaneous move game with both coaches making their decisions without knowledge of the other's strategy. To calculate the probability of success for the offense, Ruminski uses League wide statistics on effective shooting percentages to determine probabilities of success for open and contested shots. He gets

Shot	Success rate
open 2pt.	62.5%
open 3pt.	50%
Contested 2pt.	35.7%
Contested 3pt.	22.8%

Using this and the 50% probability of winning in overtime for each team, we can figure out the probability of winning for each team in all four scenarios using the following tree diagram:



(a) Use the above percentages to fill in the probabilities where appropriate on the tree diagram above.



(b) Use those probabilities to fill in the probabilities of a win for the row player (offense) in the payoff matrix below. (Note the probability for a win for the defense team is $1 - \text{prob. win for offense}$.)

	Defending Team	
	Defend 2	Defend 3
Offense	Shoot 2	Shoot 3

What are the Offense's chances of winning if they try for 2 against the inside defense? This situation starts at the left node of row 2. The offense wins with probability $0.357 \cdot 0.5 = 0.1785$. (You can calculate that the defense wins with probability $0.643 + 0.357 \cdot 0.5 = 0.8215 = 1 - 0.1785$ but this number is not needed.)

What are the Offense's chances of winning if they try for 2 against the outside defense? This situation starts at the second node of row 2. The offense wins with probability $0.625 \cdot 0.5 = 0.3125$.

What are the Offense's chances of winning if they try for 3 against the inside defense? This situation starts at the third node of row 2. The offense wins with probability 0.5.

What are the Offense's chances of winning if they try for 3 against the outside defense? This situation starts at the right hand node of row 2. The offense wins with probability 0.228.

	D2	D3
S2	0.1785	0.3125
S3	0.5	0.228

Old Exam Question

1 Rudolph (R) and Comet (C) play a game. They both choose a number between 1 and 4 simultaneously. Comet gives Rudolph a number of carrots equal to the sum of the two numbers chosen minus three. If this number is negative, Comet receives carrots from Rudolph. Which of the following give the pay-off matrix for Rudolph?

(a)

		C			
	No.	1	2	3	4
R	1	-1	0	1	2
	2	0	1	2	3
	3	1	2	3	4
	4	2	3	4	5

(b)

		C			
	No.	1	2	3	4
R	1	3	2	1	2
	2	2	1	2	3
	3	1	2	3	4
	4	2	3	4	5

(c)

		C			
	No.	1	2	3	4
R	1	2	3	4	5
	2	1	0	-1	-2
	3	0	1	2	3
	4	-1	0	1	2

(d)

		C			
	No.	1	2	3	4
R	1	-1	0	2	2
	2	0	0	2	0
	3	1	2	3	3
	4	2	3	4	4

(e)

		C			
	No.	1	2	3	4
R	1	3	4	5	6
	2	4	5	6	7
	3	5	6	7	8
	4	6	7	8	9

Application, analogous to Rock-Paper-Scissors

The biologists B. Sinervo and C. M. Lively wrote a report on a lizard species whose males are divided into three classes according to their mating behavior. Each male of the side-blotched lizards (*Uta Stansburiana*) exhibits one of three (genetically transmitted) behaviors:

- a) *highly aggressive* with a large territory that includes several females;
- b) *aggressive* with a smaller territory that holds one female;
- c) nonaggressive *sneaker* with no territory who copulates with the others' females.

In a confrontation, the highly aggressive male has an advantage over the monogamous one who in turn has an advantage over the sneaker. However, because the highly aggressive males must split their time between their various consorts, they are vulnerable to sneakers. The observed consequence of this is that the male populations cycle from a high frequency of aggressives to a high frequency of highly aggressives, then on to a high frequency of sneakers and back to a high frequency of aggressives.

Reference

Sinervo, B. and Lively, C. M., The Rock-Paper-Scissors Game and the evolution of alternative male strategies, *Nature*, **380**(1996), 240-243.